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On the Tensor Product of a Finite and an Infinite Dimensional Representation

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There are two main results in the paper. The first gives the infinitesimal character that can occur in the tensor product $V \otimes V_\lambda$ of an irreducible finite dimensional representation V_λ and an irreducible infinite dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$. The statement is that the infinitesimal characters are $\chi_{\nu+\mu_i}$, $i = 1, 2, \ldots, k$, where μ_i are the weights of V_λ and ν is the "pseudo" highest weight of V.

The second result proves that if V is a Harish-Chandra module (one which comes from a group representation), then $V \otimes V_\lambda$ has a finite composition series. But then the irreducible components in the composition series have the infinitesimal characters given in the first results.

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We are concerned here with questions about the nature of the tensor product representation

$$\pi \otimes \pi_\lambda : \mathfrak{g} \to \operatorname{End} V \otimes V_\lambda , \tag{1.1.1}$$

where $\pi_\lambda : \mathfrak{g} \to \operatorname{End} V_\lambda$ is a finite dimensional irreducible representation, say with highest weight λ, and $\pi : \mathfrak{g} \to \operatorname{End} V$ is any (finite or infinite dimensional) representation about which we have certain information.

In certain special cases a considerable amount is known about (1.1.1). For example, if V is a Verma module (a certain infinite dimensional module with a highest weight vector), then a rather complete description of $\pi \otimes \pi_\lambda$ has been given by Gelfand, Gelfand, and Bernstein [1]. Also, if $\mathfrak{g}$ is the Lie algebra of $SL(2, \mathbb{C})$, π_λ is a four-dimensional representation of $\mathfrak{g}$, and π arises from certain unitary representations of $SL(2, \mathbb{C})$ then physicists are aware of the decomposition of $V \otimes V_\lambda$ into 4-components. Along the lines of possible further applications to physics one might expect, as suggested by I. E. Segal, that a finite mass spectrum will be obtained by decomposing $V \otimes V_\lambda$ for the case

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where $\mathfrak{g}$ is the complexification of the Lie algebra of SO(3, 2) and π and π_λ are suitably chosen.

On mathematical grounds a new motive for studying (1.1.1) arises from a theorem of Lepowsky and Wallach. They have shown, using Harish–Chandra's subquotient theorem, that if $\mathfrak{g}$ is the complexification of the Lie algebra of a linear Lie group G_0 and $\mathfrak{k}_0$ is the Lie algebra of a maximal compact subgroup of G_0 then the $\mathfrak{k}_0$-finite part of any irreducible representations of G_0 occurs in the composition series of $V \otimes V_\lambda$, where π is the $\mathfrak{k}_0$-finite part of an element of the non-unitary, spherical principal series.

1.2. Regard π, π_λ, and $\pi \otimes \pi_\lambda$ as representations, also, of the universal enveloping algebra $U = U(\mathfrak{g})$ of $\mathfrak{g}$. Let Z be the center of U. One says that the arbitrary representation π admits an infinitesimal character and χ_π is that character if, for every $u \in Z$, $\pi(u)$ reduces to a scalar operator $\chi_\pi(u)$ on V. For example, it is a theorem of Dixmier that any irreducible representation π admits an infinitesimal character. Under certain conditions an infinitesimal character is a good invariant of π. If one restricts oneself to irreducible π such that every element in V is $\mathfrak{k}_0$-finite (using the notation above), then it is a theorem of Harish–Chandra that there is only a finite number of such π with the same infinitesimal character.

One of the results here is that if π is arbitrary but admits an infinitesimal character χ_π then one can write down all the possible infinitesimal characters occurring in $V \otimes V_\lambda$ explicitly in terms of χ_π and λ. In order to be explicit we have to recall how one parameterizes all the characters of Z. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{n}$ the nil radical of $\mathfrak{b}$. We regard the enveloping algebra $U(\mathfrak{h})$ of $\mathfrak{h}$ also as the ring of polynomial functions on the dual $\mathfrak{h}'$ to $\mathfrak{h}$. Let $\rho \in \mathfrak{h}'$ be defined by $\langle \rho, x\rangle = \frac{1}{2} \operatorname{tr} ad\, x \mid \mathfrak{n}$ for $x \in \mathfrak{h}$ and let $\tilde{W}$ be the group (the translated Weyl group) of all affine transformations $\tilde{\sigma}$ of $\mathfrak{h}'$ of the form $\tilde{\sigma}(\mu) = \sigma(\rho + \mu) - \rho$, where σ is in the Weyl group and $\mu \in \mathfrak{h}'$. Let $U(\mathfrak{h})^{\tilde{W}}$ be the $\tilde{W}$-invariant polynomial functions on $\mathfrak{h}'$. One then knows (a theorem of Harish–Chandra and Dynkin) that if $u \in Z$ then there exists a unique element $f_u \in U(\mathfrak{h})$ such that

$$u - f_u \in U\mathfrak{n}, \tag{1.2.1}$$

where $U\mathfrak{n}$ is the left ideal generated in U by $\mathfrak{n}$. Furthermore, $f_u \in U(\mathfrak{h})^{\tilde{W}}$ and the map

$$Z \to U(\mathfrak{h})^{\tilde{W}} \tag{1.2.2}$$

is an algebra isomorphism.

For any $\nu \in \mathfrak{h}'$ one thus obtains a character χ_ν on Z by putting, for any $u \in Z$,

$$\chi_\nu(u) = f_u(\nu). \tag{1.2.3}$$

It then follows that every character on Z is of the form χ_ν and $\chi_\nu = \chi_\mu$ if and only if ν and μ are $\tilde{W}$-conjugate. One thus parameterizes all the characters on Z by points in $\mathfrak{h}'$, or rather by the $\tilde{W}$-orbits in $\mathfrak{h}'$. See [3, Section 23.3].

Remark 1.2.1. If (μ, ν) denotes the inner product in $\mathfrak{h}'$ defined by the Killing form and we put $|\mu|^2 = (\mu, \mu)$ then the Casimir element $\mu \in Z$ is characterized by the property that for any $\nu \in \mathfrak{h}'$ one has

$$\chi_\nu(u) = |\rho + \nu|^2 - |\rho|^2. \tag{1.2.4}$$

Now return to our fixed finite dimensional representation π_λ. We may regard λ as the highest weight with respect to $\mathfrak{b}$ so that we can take $\lambda \in \mathfrak{h}'$. Let $\Delta_\lambda = \{\mu_1, \mu_2, \ldots, \mu_k\} \subseteq \mathfrak{h}'$ be the set of all the distinct weights of π_λ. Assume π (the arbitrary representation) admits an infinitesimal character χ_π. From above there exists $\nu \in \mathfrak{h}'$, unique modulo the action of $\tilde{W}$, such that $\chi_\pi = \chi_\nu$. Now consider the sequence of k characters

$$\chi_{\nu+\mu_1}, \chi_{\nu+\mu_2}, \ldots, \chi_{\nu+\mu_k}. \tag{1.2.5}$$

One observes first of all that apart from the order, (1.2.5) depends only on χ_π and not on ν. That is, if $\tilde{\sigma}\nu$ were substituted for ν in (1.2.5) we would obtain only a permutation of (1.2.5).

The result mentioned above states that if $u \in Z$ the only possible eigenvalues or in fact generalized eigenvalues of $(\pi \otimes \pi_\lambda)(u)$ are the scalars of the form $\chi_{\nu+\mu_i}(u)$, $i = 1, 2, \ldots, k$. More precisely, one has

THEOREM 5.1. *Let $u \in Z$ be arbitrary and let $\tilde{u} = (\pi \otimes \pi_\lambda)(u)$. Then the operator $\tilde{u}$ on $V \otimes V_\lambda$ satisfies the equation*

$$\prod_{i=1}^{k} (\tilde{u} - \chi_{\nu+\mu_i}(u)) = 0.$$

Remark 1.2.2. Thus if u is the Casimir element the only possible eigenvalues of u on $V \otimes V_\lambda$ are of the form $|\rho + \nu + \mu_i|^2 - |\rho|^2$, for $i = 1, 2, \ldots, k$.

In case V is a Verma module Theorem 5.1 follows from [1]. The general case can be proved using [1] together with a result of Duflo and Dixmier which determines Ker π in case V is a Verma module.

Indeed the Duflo–Dixmier result implies that

$$\operatorname{Ker} \pi \otimes \pi_\lambda \subseteq \operatorname{Ker} \pi_1 \otimes \pi_\lambda ,$$

where $\pi_1 \colon U \to \operatorname{End} V_1$ is an arbitrary representative admitting the same infinitesimal character as π. In this paper, however, Theorem 5.1 will be proved by another method, which I think has independent interest.

One can deal with Z all at once. Combining Theorem 5.3 with Corollary 5.6, one has

THEOREM 1.2. *Assume $X_1 \subsetneq X_2 \subseteq V \otimes V_\lambda$ are U-submodules and the representation of U on X_2/X_1 admits a character χ. Then χ is necessarily of the form $\chi = \chi_{\nu+\mu_i}$ for some $i = 1, 2,\ldots, k$. Furthermore, put $P_0 = 0$ and*

$$P_i = \left\{ y \in V \otimes V_\lambda \;\middle|\; \prod_{j=1}^{i} (\tilde{u} - \chi_{\nu+\mu_j}(u))\, y = 0 \text{ for all } u \in Z \right\}$$

so that

$$0 = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_k = V \otimes V_\lambda$$

is a filtration of $V \otimes V_\lambda$ by U-submodules. Then if $P_i \mid P_{i-1} \neq 0$ it admits an infinitesimal character and that character is just $\chi_{\nu+\mu_i}$.

For a Zariski open set $\mathfrak{h}_0'$ in $\mathfrak{h}'$ the characters $\chi_{\nu+\mu_i}$, $i = 1, 2,\ldots, k$, are distinct for $\nu \in \mathfrak{h}_0'$ (i.e., $\nu + \mu_i$ and $\nu + \mu_j$ are not $\tilde{W}$-conjugate for $i \neq j$). In this case one obtains a direct sum.

COROLLARY 5.5. *If the characters $\chi_{\nu+\mu_i}$, $i = 1, 2,\ldots, k$, are distinct and we put*

$$Y_i = \{ y \in V \otimes V_\lambda \mid \tilde{u} y = \chi_{\nu+\mu_i}(u)\, y \text{ for all } u \in Z \}$$

so that, if not zero, Y_i is the maximal submodule of $V \otimes V_\lambda$ which admits the infinitesimal character $\chi_{\nu+\mu_i}$, then

$$V \otimes V_\lambda = \bigoplus_{i=1}^{k} Y_i .$$

1.3. In the results of Section 1.2 nothing was stated about irreducible submodules of $V \otimes V_\lambda$. In fact even if π is irreducible we do not know whether $V \otimes V_\lambda$ has a finite composition series. However, if π is irreducible and V is a Harish-Chandra module, that is, V is $\mathfrak{k}_0$-finitely semisimple, then we prove that $V \otimes V_\lambda$ does indeed have a finite composition series. Thus the characters given in Section 1.2 are characters of the irreducible module occurring in the composition series in $V \otimes V_\lambda$. The assumption that V is a Harish–Chandra module of course is satisfied when V arises from a group representation. The proof that $V \otimes V_\lambda$ has a finite composition series is ring theoretic and exploits the fact that U is noetherian.

2. Preliminaries on Infinitesimal Characters

2.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $\mathfrak{h}'$ the dual space to $\mathfrak{h}$, $\Delta \subseteq \mathfrak{h}'$ the set of roots with respect to $(\mathfrak{h}, \mathfrak{g})$, and $\Delta_+ \subseteq \Delta$ a system of positive roots.

Let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and $U(\mathfrak{h}) \subseteq U$ the enveloping algebra of $\mathfrak{h}$. We will also regard $U(\mathfrak{h})$ as the ring of polynomial functions on $\mathfrak{h}'$, the dual space to $\mathfrak{h}$. Thus the group A of invertible affine transformations of $\mathfrak{h}'$ operates, as a group of automorphisms, on $U(\mathfrak{h})$ where if $\tau \in A$ and $u \in U(\mathfrak{h})$, $\lambda \in \mathfrak{h}'$, then $(\tau u)(\lambda) = u(\tau^{-1}\lambda)$. In particular the Weyl group W, corresponding to $(\mathfrak{h}, \mathfrak{g})$, operates on $U(\mathfrak{h})$. Similarly, the "translated Weyl group $\tilde{W}$" operates on $U(\mathfrak{h})$. We recall the definition of $\tilde{W}$. Let $\rho = \frac{1}{2}\sum_{\varphi \in \Delta_+} \in \mathfrak{h}'$ and for any $\nu \in \mathfrak{h}'$ let $\tau_\nu \colon \mathfrak{h}' \to \mathfrak{h}'$ be the translation map given by $\tau_\nu(\lambda) = \lambda + \nu$. Clearly $\tau_\nu \in A$. Then by definition $\tilde{W}$ is the conjugate $\tau_{-\rho} W \tau_\rho$ of W in A. That is, $\tilde{W}$ is the group of all elements in A of the form $\tilde{\sigma}$, where $\sigma \in W$ and $\tilde{\sigma} = \tau_{-\rho}\sigma\tau_\rho$. One then has

$$\sigma(\lambda) = \sigma(\lambda + \rho) - \rho \tag{2.1.1}$$

for any $\lambda \in \mathfrak{h}'$, or if $u \in U(\mathfrak{h})$ then $\tilde{\sigma}^{-1}u \in U(\mathfrak{h})$ is given by

$$(\tilde{\sigma}^{-1}u)(\lambda) = u(\sigma(\lambda + \rho) - \rho). \tag{2.1.2}$$

Remark 2.1. One notes that $-\rho$, the "translated origin," is the unique element in $\mathfrak{h}'$ fixed under all the elements in $\tilde{W}$.

2.2. Now let Z be the center of U. Also let $\mathfrak{n} \subseteq \mathfrak{g}$ be the nilpotent subalgebra spanned by all root vectors $e_\varphi \in \mathfrak{g}$ corresponding to roots $\varphi \in \Delta_+$. For any $u \in Z$ one knows, then, that there is a unique element $f_u \in U(\mathfrak{h})$ such that

$$u - f_u \in U\mathfrak{n} \tag{2.2.1}$$

($U\mathfrak{n}$ is the left ideal in U generated by $\mathfrak{n}$). Also the map $Z \to U(\mathfrak{h})$, $u \mapsto f_u$ is a homomorphism of algebras (see, e.g., [3, p. 130]). In fact one has the following result of Dynkin and, independently, of Harish-Chandra. (If a group G operates on a set S then $S^G \subseteq S$ denotes the subset of invariant elements in S. A similar notation is used for a Lie algebra operating on a vector space.)

THEOREM 2.2. *For any $u \in Z$ one has $f_u \in U(\mathfrak{h})^{\tilde{W}}$ and the map*

$$Z \to U(\mathfrak{h})^{\tilde{W}},$$

$u \mapsto f_u$, is an algebra isomorphism.

Remark 2.2. Note that (using (2.1.2)) if $g \in U(\mathfrak{h})$ then $g \in U(\mathfrak{h})^{\tilde{W}}$ if and only if $g = \tilde{f}$ for some $f \in U(\mathfrak{h})^W$, where $\tilde{f} \in U(\mathfrak{h})$ is defined by

$$\tilde{f}(\lambda) = f(\lambda + \rho) \tag{2.2.2}$$

for any $\lambda \in \mathfrak{h}'$.

2.3. Now if $l = \operatorname{rank} \mathfrak{g} = \dim \mathfrak{h}$ then one knows there exist $f_1, \ldots, f_l \in U(\mathfrak{h})^W$ such that $U(\mathfrak{h})^W$ is the algebra generated by the f_i over $\mathbb{C}$, i.e., in the usual notation of commutative ring theory,

$$U(\mathfrak{h})^W = \mathbb{C}[f_1, \ldots, f_l], \tag{2.2.3}$$

and also that any such $f_1, \ldots, f_l$ are necessarily algebraically independent. It follows therefore that $U(\mathfrak{h})^{\tilde{W}} = C[\tilde{f}_1, \ldots, \tilde{f}_l]$ and the $\tilde{f}_i$ are algebraically independent.

Hence as a consequence of Theorem 2.2 one has

COROLLARY 2.3. *One has $Z = \mathbb{C}[u_1, \ldots, u_l]$, where $u_i \in Z$ is defined by*

$$f_{u_i} = \tilde{f}_i$$

and where $f_1, \ldots, f_l \in U(\mathfrak{h})^W$ satisfy (2.2.3). *Furthermore, the u_i are algebraically independent.*

Remark 2.3. The Casimir element $u \in Z$ defined by the Killing form is characterized by the property that

$$f_u(\lambda) = |\rho + \lambda|^2 - |\rho|^2$$

for any $\lambda \in \mathfrak{h}'$. (See, e.g., [6, Proposition 5.6].) Here $|\nu|^2 = (\nu, \nu)$ for $\nu \in \mathfrak{h}'$, where the inner product on $\mathfrak{h}'$ is induced by the Killing form.

2.4. A character χ of Z, here, will simply mean an algebra homomorphism χ of Z into the scalars $\mathbb{C}$. Let $\hat{Z}$ be the set of all characters of Z.

Now if $u_1, \ldots, u_l \in Z$ are algebraically independent, as in Corollary 2.3, then since they generate Z, a character $\chi \in \hat{Z}$ is determined by the values $\chi(u_i)$. Furthermore, since the u_i are algebraically independent, these values are arbitrary. Thus if $c = (c_1, \ldots, c_l) \in \mathbb{C}^l$ there exists a unique character $\chi_c \in \hat{Z}$ such that

$$c_i = \chi_c(u_i). \tag{2.4.1}$$

That is, one has

COROLLARY 2.4. *The map*

$$\mathbb{C}^l \to \hat{Z} \tag{2.4.2}$$

defined by $c \mapsto \chi_c$ *is a bijection.*

2.5. For any $\lambda \in \mathfrak{h}'$ one clearly defines Theorem 2.2 using a character $\chi_\lambda \in \hat{Z}$ by putting

$$\chi_\lambda(u) = f_u(\lambda). \tag{2.5.1}$$

The correspondence $\lambda \mapsto \chi_\lambda$ defines a map

$$\mathfrak{h}' \to \hat{Z}. \tag{2.5.2}$$

An easy consequence of Theorem 2.2 is the following result of Harish–Chandra.

THEOREM 2.5. *The map* (2.5.2) *is surjective. Furthermore,* $\chi_\lambda = \chi_\nu$ *if and only if* λ *and* ν *are* $\tilde{W}$*-conjugate.*

Proof. Using Theorem 2.2 it suffices to show that any homomorphism of $U(\mathfrak{h})^{\tilde{W}}$ into $\mathbb{C}$ is obtained by evaluating the elements $f \in U(\mathfrak{h})^{\tilde{W}}$ at some point $\lambda \in \mathfrak{h}'$ and that two points define the same homomorphism if and only if they are $\tilde{W}$-conjugate. But this is a special case of the theorem that if G is a reductive algebraic group operating on an affine variety X, say over $\mathbb{C}$, and $A(X)$ denotes the affine ring of X, then any homomorphism into $\mathbb{C}$ of $A(X)^G$ is defined by evaluating the elements of $A(X)^G$ on a Zariski closed G-orbit in X and this sets up a bijective correspondence between all Zariski closed G-orbits and all homomorphisms of $A(X)^G$ into $\mathbb{C}$ (see, e.g., [2]). In the case at hand, since $\tilde{W}$ is finite, all orbits are Zariski closed.

Q.E.D.

Another proof of part of Theorem 2.5 is given in [3] (see also [3, Exercise 9, p. 134]).

Remark 2.5. If X denotes the set of all orbits of $\tilde{W}$ in $\mathfrak{h}'$, then Theorem 2.5 sets up a bijection

$$X \to \hat{Z}. \tag{2.5.3}$$

One notes that (2.5.3) is invariantly defined whereas (2.4.2) depends upon the choice of $u_1, ..., u_l \in Z$ (or rather $f_1, ..., f_l \in U(\mathfrak{h})^W$ satisfying (2.2.3)).

2.6. Let

$$\pi: U \to \operatorname{End} V$$

be any representation of U on a complex vector space (finite or infinite dimensional). We say that π admits an infinitesimal character if for any $u \in Z$ there exists a scalar $\chi_\pi(u) \in \mathbb{C}$ such that

$$\pi(u) = \chi_\pi(u)\, 1_V,$$

where 1_V is the identity operator on V. In such a case it is obvious that $u \to \chi_\pi(u)$ is a character on Z and χ_π is called the infinitesimal character of π.

It is a theorem of Dixmier's that every irreducible representation of U admits an infinite character. Special cases of such representations are the representations π_λ, $\lambda \in \mathfrak{h}'$, of U. One knows (see [3]) that for any $\lambda \in \mathfrak{h}'$ there exists a unique, up to equivalence, irreducible representation

$$\pi_\lambda: U \to \operatorname{End} V_\lambda$$

having a highest-weight vector v_λ with weight λ (i.e., $0 \neq v_\lambda \in V_\lambda{}^{\mathfrak{n}}$ and $x \cdot v_\lambda = \langle \lambda, x \rangle v_\lambda$ for all $x \in \mathfrak{h}$). One significance of the map (2.5.2) is

PROPOSITION 2.6. *Let $\lambda \in \mathfrak{h}'$ be arbitrary. Then*

$$\chi_{\pi_\lambda} = \chi_\lambda .$$

Proof. Since the left ideal $U\mathfrak{n}$ in U annihilates the highest-weight vector $v_\lambda \in V_\lambda$, one has, for any $u \in Z$, $\pi_\lambda(u)v_\lambda = \pi_\lambda(f_u)v_\lambda$. But clearly $\pi_\lambda(f_u)v_\lambda = f_u(\lambda)v_\lambda$. Thus $\pi_\lambda(u)$ takes the scalar value $f_u(\lambda)$ on v_λ and hence on the whole vector space V_λ. That is, $\chi_{\pi_\lambda}(u) = f_u(\lambda) = \chi_\lambda(u)$. Q.E.D.

Remark 2.5. If $u \in Z$ is the Casimir element then Proposition 2.5 recovers the well-known formula

$$\chi_{\pi_\lambda}(u) = |\rho + \lambda|^2 - |\rho|^2.$$

(See Remark 2.3.)

3. On Noetherian and Artinian Conditions in Tensor Products

3.1. The notion of noetherian rings and noetherian modules applies to noncommutative rings as well as to the more familiar commutative ring situation. A module M for a not necessarily commutative ring R (with unit) is noetherian if every submodule of M is finitely generated over R, or equivalently, if every collection of submodules has a maximal element. R itself is noetherian if as a module over itself, by left multiplication, it is noetherian; that is, if for any increasing sequence of left ideals $I_n \subseteq R$ there exists an integer N such that $I_n = I_N$ for all $n \geqslant N$.

One then knows, and easily establishes, that any finitely generated module M over a noetherian ring is noetherian.

A module M over R is said to have a finite composition series if we can find a finite sequence of submodules

$$0 = M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_n = M$$

such that M_i/M_{i-1}, $i = 1, 2, \ldots, n$, is an irreducible R-module.

In such a case the sequence of submodules M_i is called a Jordan–Holder sequence and n is its length. Furthermore, one knows that any other Jordan–Holder sequence M_i' in M has length n and apart from the order the irreducible representations of R on M_i/M_{i-1} and M_i'/M_{i-1}', $i = 1, 2, \ldots, n$, are the same, including multiplicities. Thus if M has a composition series and ν is some irreducible representation of R it makes sense to speak of the multiplicity of ν in M. One also knows that if $0 = N_0 \subsetneqq N_1 \subsetneqq \cdots \subsetneqq N_k = M$ is a sequence of submodules then $k \leqslant n$ and the N_i can be embedded in a Jordan–Holder sequence.

A module M of R is called artinian if every set of submodules has a minimal element. One knows

Proposition 3.1. *Let M be an R-module. Then M has a finite composition series if and only if it is both noetherian and artinian.*

3.2. The considerations in Section 3.1 will apply to the enveloping algebra $U = U(\mathfrak{g})$ since first of all one has

PROPOSITION 3.2. *U is noetherian.*

Proposition 3.2 is an easy consequence of the Birkhoff–Witt theorem and the fact that the symmetric algebra $S(\mathfrak{g})$, a polynomial ring in $\dim \mathfrak{g}$ generators, is noetherian. Using the natural filtration in U, every left ideal I in U defines an ideal $\bar{I}$ in $S(\mathfrak{g})$ and $I \not\subset J$ if and only if $\bar{I} \not\subset \bar{J}$ in $S(\mathfrak{g})$.

3.3. Now $U \otimes U$ is an algebra and one has a homomorphism (the diagonal homomorphism)

$$d: U \to U \otimes U, \tag{3.3.1}$$

where if $x \in \mathfrak{g}$ then $dx = x \otimes 1 + 1 \otimes x$. In general du for an arbitrary $u \in U$ is a more complicated expression which may be written

$$du = \sum_{i=1}^{k} u_i' \otimes u_i'', \tag{3.3.2}$$

where $u_i', u_i'' \in U$. For example, if x_i, $i = 1, 2, \ldots, n$, is an orthonormal basis of $\mathfrak{g}$, say with respect to the Killing form in $\mathfrak{g}$, and $u = \sum_{i=1}^{n} x_i^2 \in Z$ is the corresponding Casimir element, one has

$$du = u \otimes 1 + 1 \otimes u + 2 \sum_{i=1}^{n} x_i \otimes x_i .$$

Now if V_1 and V_2 are two U-modules then the tensor product $V = V_1 \otimes V_2$ is a U-module, where if $v_i \in V_i$, $i = 1, 2$, and $x \in \mathfrak{g}$ then $x(v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2$. More generally, if $u \in U$ and du is as in (3.3.2) then

$$u(v_1 \otimes v_2) = \sum_{i=1}^{k} u_i'v_1 \otimes u_i''v_2 .$$

Some time ago we observed that if V_2 is finite dimensional and V_1 is noetherian and satisfies a condition similar to that stated in Section 3.4 then $V_1 \otimes V_2$ is again noetherian. However, it was pointed out to me by G. W. McCollum that my argument was needlessly complicated and the latter assumption on V_1 is unnecessary. The following more general statement and very simple proof is due to McCollum.

PROPOSITION 3.3. *If V_1 is noetherian and V_2 is finite dimensional then the U-module $V_1 \otimes V_2$ is again noetherian.*

Proof. Now since V_1 is noetherian there exists a finite dimensional subspace $Y_1 \subseteq V_1$ such that $UY_1 = V_1$.

But since U is noetherian (Proposition 3.2), to prove that $V_1 \otimes V_2$ is noetherian it suffices to prove only that $V_1 \otimes V_2$ is finitely generated. In fact we will show that

$$U(Y_1 \otimes V_2) = V_1 \otimes V_2 .$$

Let $W = U(Y_1 \otimes V_2)$ and let $\tilde{V}_1 = \{v \in V_1 \mid v \otimes V_2 \subseteq W\}$. It clearly suffices to show that $\tilde{V}_1 = V_1$. But since $Y_1 \subseteq \tilde{V}_1$ it just suffices to show that $\tilde{V}_1$ is stable under $\mathfrak{g}$. Let $v_1 \in \tilde{V}_1$, $v_2 \in V_2$, and $x \in \mathfrak{g}$. We have only to show that $xv_1 \otimes v_2 \in W$. But $xv_1 \otimes v_2 = x(v_1 \otimes v_2) - v_1 \otimes xv_2$. However, $v_1 \otimes xv_2$ and $v_1 \otimes v_2$ both are in W since $v_1 \in \tilde{V}_1$. But also $x(v_1 \otimes v_2) \in W$ since W is $\mathfrak{g}$-stable. Thus $xv_1 \otimes v_2 \in W$. Q.E.D.

3.4. A subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is called reductive in $\mathfrak{g}$ if $\mathfrak{g}$ is a completely reducible module under the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$. We assume that $\mathfrak{k}$ is such a subalgebra fixed throughout. Let $\hat{\mathfrak{k}}$ denote the set of equivalence classes of all finite dimensional irreducible representations of $\mathfrak{k}$.

A U-module V will be called $\mathfrak{k}$-finitely semisimple if V can be written as a sum of finite dimensional irreducible $\mathfrak{k}$-modules and each $\gamma \in \hat{\mathfrak{k}}$ occurs with finite multiplicity in V. Two examples are: (1) $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$ and $V = V_\lambda$ for some $\lambda \in \mathfrak{h}'$, in the notation of Section 2.6.

(2) Assume G_0 is a semisimple Lie group with a finite center such that $\mathfrak{g}$ is the complexification of its Lie algebra. Here let $\mathfrak{k}$ be the complexification of the Lie algebra of a maximal compact subgroup of G_0. Then by a theorem of Harish-Chandra any irreducible Banach space B representation of G_0 gives rise to a $\mathfrak{k}$-finitely simple U-module V by putting V equal to the space of all smooth vectors in B which are $\mathfrak{k}$-finite. Incidentally, one knows that all the elements in V are analytic and that V is dense in B.

Assume that V is a $\mathfrak{k}$-finitely semisimple U-module. Thus we may write as a direct sum

$$V = \bigoplus_{\gamma \in \hat{\mathfrak{k}}} V_\gamma , \tag{3.4.1}$$

where V_γ, $\gamma \in \hat{\mathfrak{k}}$, is the set of all $v \in V$ which transform under $\mathfrak{k}$

according to the representation γ. By assumption, V_γ is finite dimensional.

Now if V^* is the dual space to V then V^* is a U-module, where if $x \in \mathfrak{g}$, $v' \in V^*$, one defines $xv' \in V^*$ so that for any $v \in V$ one has

$$\langle xv', v\rangle = -\langle v', xv\rangle.$$

Now let $V' \subseteq V^*$ be the subspace of all $\mathfrak{k}$-finite elements. Since any element $u \in U$ is $\mathfrak{k}$-finite under the adjoint action of u on U it is easy to see that V' is a U-submodule of V^*. In fact, using the direct sum decomposition (3.4.1) of V we may identify V^* with the direct product of the dual spaces $(V_\gamma)'$ over all $\gamma \in \hat{\mathfrak{k}}$. It then follows easily, using say, the Chinese remainder theorem for $U(\mathfrak{k})$, that V' is the set of all elements in V^* with only a finite number of components different from zero. It then follows that V' is $\mathfrak{k}$-finitely semisimple and if the primary decomposition of V' is written as

$$V' = \bigoplus_{\gamma\in\hat{\mathfrak{k}}} V'_{\gamma'}, \tag{3.4.2}$$

where $\gamma' \in \hat{\mathfrak{k}}$ is the representation class contragredient to γ, then $V'_{\gamma'}$ is nonsingularly paired to V_γ, so that $\dim V_\gamma = \dim V'_{\gamma'}$; and $V'_{\gamma'}$ may be characterized as the set of all $v' \in V^*$ which are orthogonal to all V_μ, where $\mu \neq \gamma$.

Now one clearly has a natural isomorphism $(V')' = V''$ with V so that we may identify $V'' = V$. For any subspace $W \subseteq V$ let $W^o \subseteq V'$ be its orthogonal complement. It is clear that if W is a U-submodule of V then W^o is a U-submodule of V'. Furthermore one easily shows, using the Chinese remainder theorem for $U(\mathfrak{k})$, that

$$W = \bigoplus_{\gamma\in\hat{\mathfrak{k}}} (W \cap V_\gamma) \tag{3.4.3}$$

and similarly,

$$W^o = \bigoplus_{\gamma\in\hat{\mathfrak{k}}} W^o \cap V'_{\gamma'}\,.$$

But then clearly $W^o \cap V'_{\gamma'}$ is the orthocomplement of $W \cap V_\gamma$ in V_γ'. It then follows easily that one has

LEMMA 3.4. *If W is a U-submodule of V then $W = W^{oo}$.*

Remark 3.4. As a consequence of Lemma 3.4 one notes that if W_1, $W_2 \subseteq V$ are submodules then $W_1 \subsetneqq W_2$ if and only if $W_2{}^0 \subsetneqq W_1{}^0$.

Remark 3.4 immediately implies

PROPOSITION 3.4. *Assume that V is a $\mathfrak{k}$-finitely semisimple U-module. Then V is noetherian (resp., artinian) if and only if V' is artinian (resp., noetherian).*

In particular (using Proposition 3.1) V has a finite composition series if and only if V' has a finite composition series.

Finally, V is U-irreducible if and only if V' is U-irreducible.

3.5. It is convenient to introduce the category whose objects $\mathscr{C}$ are $\mathfrak{k}$-finitely semisimple U-modules V which admit a finite composition series. The morphisms are just U-module homomorphisms. It is clear that if $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of U-modules then $V \in \mathscr{C}$ if and only if both $V_1, V_2 \in \mathscr{C}$. By Proposition 3.4, for any object $V \in \mathscr{C}$ one has a dual object V' and $V \to V'$ is a contravariant functor. We also note that all finite dimensional U-modules are in $\mathscr{C}$. The main point, however, is that $\mathscr{C}$ is closed under tensor products with finite dimensional modules.

THEOREM 3.5. *Assume V_2 is any finite dimensional U-module and that V_1 is a $\mathfrak{k}$-finitely semisimple U-module admitting a finite composition series. Then $V = V_1 \otimes V_2$ is again a $\mathfrak{k}$-finitely semisimple module admitting a finite composition series.*

Proof. If $\gamma, \beta \in \hat{\mathfrak{k}}$ it is easy to see that there are only a finite number of elements $\delta \in \hat{\mathfrak{k}}'$ such that β occurs in the tensor product decomposition of γ and δ with nonzero multiplicity. Indeed, by Schur's lemma, β occurs in the tensor product of γ and δ if and only if the identity representation occurs in the triple tensor product of β', γ, and δ. But again by Schur's lemma this occurs if and only if δ' occurs in the tensor product of β' and γ. Since the latter is finite dimensional only a finite number of δ have the property.

Thus since only a finite number of γ occur in V_2 this implies that any fixed β occurs with finite multiplicity in $V_1 \otimes V_2$. In fact the argument above shows that there are only a finite number of δ such that β occurs in $(V_1)_\delta \otimes V_2$. But $\dim(V_1)_\delta \otimes V_2$ is finite dimensional. Thus $V_1 \otimes V_2$ is $\mathfrak{k}$-finitely semisimple. Similarly $V_1' \otimes V_2'$ is $\mathfrak{k}$-finitely semisimple.

But $V_1 \otimes V_2$ and $V_1' \otimes V_2'$ are paired, where

$$\langle v_1' \otimes v_2', v_1 \otimes v_2 \rangle = \langle v_1', v_1 \rangle \langle v_2', v_2 \rangle \tag{3.5.1}$$

for $v_i \in V_i$, $v_i' \in V_i'$, $i = 1, 2$.

The pairing (3.5.1) induces a linear map $\tau: V_1' \otimes V_2' \to (V_1 \otimes V_2)^*$. However, since τ is clearly a U-module map and since $V_1' \otimes V_2'$ is

$\mathfrak{k}$-finitely semisimple, the image of τ is in $(V_1 \otimes V_2)'$ and hence the pairing (3.5.2) induces a U-module map

$$V_1' \otimes V_2' \to (V_1 \otimes V_2)'. \tag{3.5.2}$$

We assert that the map (3.5.2) is an isomorphism. Indeed to prove this it suffices to show that the pairing (3.5.1) is nonsingular, which in fact implies that the map (3.5.2) is injective. But it is surjective since if the image W is not equal to $(V_1 \otimes V_2)'$ one has $W^o \subseteq V_1 \otimes V_2$ is not zero, by Lemma 3.4. But W^o is orthogonal to $V_1' \otimes V_2'$, contradicting the nonsingularity of the pairing.

By symmetry, to prove that the pairing (3.5.1) is nonsingular it is enough to show if $0 \neq t = \sum_{i=1}^{k} u_i \otimes w_i \in V_1 \otimes V_2$, where $u_i \in V_1$ and $w_i \in V_2$, there exists $s \in V_1' \otimes V_2'$ such that $\langle s, t\rangle \neq 0$. Since $t \neq 0$ we may assume that the w_i are linearly independent and the u_i are nonzero. But then we may find $w' \in V_2'$ such that $\langle w', w_i\rangle = 0$ for $i > 1$ and $\langle w', w_1\rangle \neq 0$, and since V_1 and V_1' are nonsingularly paired we may find $u' \in V_1$ such that $\langle u', u_1\rangle \neq 0$. Thus $\langle s, t\rangle = \langle u', u_1\rangle\langle w', w_1\rangle \neq 0$. Thus (3.5.2) is an isomorphism.

But now by Proposition 3.3, $V_1 \otimes V_2$ is noetherian and hence $(V_1 \otimes V_2)'$ is artinian by Proposition 3.4. However, by Proposition 3.3, $V_1' \otimes V_2'$ is also noetherian. Since (3.5.2) is an isomorphism this implies that $(V_1 \otimes V_2)'$ is also noetherian. Thus $(V_1 \otimes V_2)'$ is both noetherian and artinian. Thus by Proposition 3.1, $(V_1 \otimes V_2)'$ has a finite composition series and hence $V_1 \otimes V_2$ has a finite composition series, by Proposition 3.4. Q.E.D.

COROLLARY 3.5. *Assume V_1 and V_2 are irreducible U-modules which are $\mathfrak{k}$-finitely semisimple. If V_2 is finite dimensional then $V_1 \otimes V_2$ has a finite composition series.*

3.6. Let V_1 and V_2 be as in Corollary 3.5 and put $M = V_1 \otimes V_2$. By Corollary 3.5 one has a Jordan–Holder sequence $0 = M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_n$. Now consider the question: Assume the representations on V_1 and V_2 are known. What can one say about the irreducible representation π_i of U on the quotients M_i/M_{i-1}, $i = 1, 2, \ldots, n$? Although we cannot determine the π_i it is the main point of the paper that we can say a good deal about the infinitesimal characters χ_{π_i} of the π_i. The character χ_{π_i} does not determine π_i. However, by a theorem of Harish–Chandra there are at most a finite number of $\mathfrak{k}$-finitely semisimple irreducible representations of $\mathfrak{g}$ with the same infinitesimal character. Thus in principle we can determine the π_i up to finite ambiguity.

Remark 3.6.1. Although it is quite speculative the knowledge of the infinitesimal characters χ_{π_i} may be of direct significance in physics. I.E. Segal suggests that if $G_0 = \mathrm{SO}(3, 2)$ in Example (2) of Section 3.4 then for suitable π, $\chi_\pi(u)$, where u is the Casimir element, may be related to the mass of an elementary particle. He further suggests that if V_1 corresponds (in the sense of (2), Section 3.4) to the unitary representation associated to a free massless particle and V_2 is, say, the spin representation of SO(3, 2) (so dim $V_1 = 4$) then the components M_i/M_{i-1} may possibly correspond to a finite grouping of particles with mass. The mass of these particles would then be related to the numbers $\chi_{\pi_i}(u)$.

Remark 3.6.2. A mathematical motivation for considering the reduction of $V_1 \otimes V_2$ when V_2 is finite dimensional and V_1 is infinite dimensional arises from the unsolved question of determining all $\mathfrak{k}$-finitely semisimple irreducible representations V of U. More restrictively, say that V is $\mathfrak{k}$-finitely semisimple in the strong sense if, for whenever $\gamma \in \hat{\mathfrak{k}}$ occurs in V, it also occurs in some finite dimensional representation of U. The point is that using Harish-Chandra's subquotient theorem, Lepowsky and Wallach have proved that every irreducible $\mathfrak{k}$-finitely semisimple module V in the strong sense occurs in the reduction of $V_1 \otimes V_2$, where V_2 is finite dimensional and V_1 belongs to the spherical nonunitary principal series. An advantage to this approach is that the spherical nonunitary principal series is a family of representations that are relatively easy to deal with. A drawback, however, is that V_1 itself is not necessarily irreducible and one must first find its composition factors. However, in the split rank 1 case this has been done by Johnson and Wallach.

4. The Minimal Polynomial Equation Satisfied by $\delta(u)$ in $U \otimes \mathrm{End}\ V_\lambda$

4.1. Recall that for any $\lambda \in \mathfrak{h}'$,

$$\pi_\lambda \colon U \to \mathrm{End}\ V_\lambda \tag{4.1.1}$$

denotes the irreducible representation of U having a highest-weight vector with weight λ.

Now one knows that V_λ is finite dimensional if and only if $\lambda \in D_Z$, where D_Z is the set of dominant integral linear forms in $\mathfrak{h}$ (see, e.g., [3]). We recall that D_Z has the structure of a finitely generated discrete semigroup in $\mathfrak{h}'$ and D_Z spans $\mathfrak{h}'$. Here we regard D_Z as partially ordered (in a strong sense) with the order relation $\mu \geqslant \nu$ in case $\mu - \nu \in D_Z$. We make use of the following easy proposition.

PROPOSITION 4.1. *Let $\nu_0 \in D_Z$ be arbitrary and let f be any polynomial function on $\mathfrak{h}'$. Then f vanishes identically on h' if and only if $f(\nu) = 0$ for all $\nu \in D_Z$, where $\nu \geqslant \nu_0$.*

Proof. We have only to observe that the set S of all $\nu \in D_Z$ such that $\nu \geqslant \nu_0$ contains a basis $\nu_1, \ldots, \nu_l$ of $\mathfrak{h}'$. But by assumption one has $f(\sum_{i=1}^{l} n_i \cdot \nu_i) = 0$ for all $n_i \in Z$, $n_i \geqslant 0$, $i = 1, \ldots, l$. Expressing f as a polynomial in the dual basis to ν_i the proof reduces to observing that a polynomial function on $\mathbb{C}^l$ vanishes in case $f(n_1, \ldots, n_l) = 0$ for all choices $n_1, \ldots, n_l$ of nonnegative integers. Q.E.D.

4.2. Now fix $\lambda \in D_Z$ for the remainder of the paper and let $d = \dim V_\lambda$. For any $\nu \in D_Z$ let $d(\nu)$ denote the number of irreducible U-components in $V_\nu \otimes V_\lambda$. In [4, Lemma 4.1] we proved that one always has $d(\nu) \leqslant d$. Furthermore, in [4] we defined what was meant by saying that π_λ is totally subordinate to π_ν and we proved that π_λ is totally subordinate to π_ν if and only if $d(\nu) = d$. For the purposes of this paper we define totally subordinate to mean that $d(\nu) = d$. But then [4, Theorem 5.1] implies

PROPOSITION 4.2. *Let $\lambda \in D_Z$; then there exists $\nu_0 \in D_Z$ such that π_λ is totally subordinate to π_ν (that is, $d(\nu) = d = \dim V_\lambda$) whenever $\nu \geqslant \nu_0$.*

4.3. In [4, theorem 5.1] there is more information than in Proposition 4.2 and some of this information will also be useful here. Let $\Delta_\lambda = \{\mu_1, \ldots, \mu_k\} \subseteq \mathfrak{h}'$ be the *set* of all weights of the representation π_λ and for each $i = 1, 2, \ldots, k$ let d_i be the multiplicity of μ_1 so that

$$d = \sum_{i=1}^{k} d_i . \tag{4.3.1}$$

Now, given $\nu \in D_Z$ let $d_i(\nu)$ be the multiplicity of the representation $\pi_{\nu+\mu_i}$ in $V_\nu \otimes V_\lambda$.

Remark 4.3.1. If $\nu + \mu_i \notin D_Z$ then $\pi_{\nu+\mu_i}$ is infinite dimensional. Hence one necessarily has $d_i(\nu) = 0$ for such a value of i.

The following result is implicit in [4, Lemma 4.1] if one considers it on a weight-by-weight basis. However, it is explicitly given in [7] (see [7, Formula (2.2.4), p. 394]).

THEOREM 4.3. *Every irreducible component on* $V_\nu \otimes V_\lambda$ *is of the form* $\pi_{\nu+\mu_i}$ *for some* $i = 1, 2,..., k$. *Furthermore one always has* $d_i(\nu) \leqslant d_i$ *so that* π_λ *is totally subordinate to* π_ν *if and only if* $d_i(\nu) = d_i$ *for* $i = 1,..., k$.

Remark 4.3.2. Note that Theorem 4.3 implies that

$$d(\nu) = \sum_{i=1}^{k} d_i(\nu).$$

4.4. Now for each $\xi \in D_Z$ let $l(\xi)$ denote the multiplicity of the zero weight in V_ξ. That is, $l(\xi) = \dim V_\xi^{\mathfrak{h}}$ where $V_\xi^{\mathfrak{h}}$ is the space of $\pi_\xi(\mathfrak{h})$ invariants.

Now U is a $\mathfrak{g}$-module with respect to the adjoint representation of $\mathfrak{g}$ in U. We recall the structure of U as a $\mathfrak{g}$-module given in [5, Theorem 21]. Let $E \subseteq U$ denote the space (set of harmonic elements) spanned by all powers x^k, where $x \in \mathfrak{g}$ is nilpotent. Then one has

$$U = Z \otimes E, \tag{4.4.1}$$

where Z is the center of U and the tensor product corresponds to multiplication. Another part of [5, Theorem 21] may be restated as

THEOREM 4.4. *E is stable under the adjoint representation* $\mathfrak{g}$ *in* U *and for any* $\xi \in D_Z$ *the representation* π_ξ *occurs with multiplicity* $l(\xi)$ *in* E.

4.5. Now let

$$\pi: U \to \operatorname{End} V \tag{4.5.1}$$

be any (possibly infinite dimensional) representation of U. We recall that one says π admits an infinitesimal character if $\pi(u)$ is a scalar for any $u \in Z$. (Thus π admits an infinitesimal character if it is irreducible.) In such a case write χ_π for the infinitesimal character so that $\pi(u) = \chi_\pi(u) \cdot 1_V$ for any $u \in Z$.

Now, given the arbitrary representation π regard End V as a $\mathfrak{g}$-module, where if $\alpha \in \operatorname{End} V$ and $x \in \mathfrak{g}$ then $x \cdot \alpha = [\pi(x), \alpha]$. It is clear, then, that the map π in (4.5.1) is a map of $\mathfrak{g}$-modules, and it follows that $\pi(U)$ is a submodule of End V; since U is completely reducible as a $\mathfrak{g}$-module the same is true of $\pi(U)$. For any $\xi \in D_Z$ let $l_\pi(\xi)$ denote the multiplicity (possibly infinite) of π_ξ in $\pi(U)$.

COROLLARY 4.5. *Assume π is any representation of U which admits an infinitesimal character. Thus for any $\xi \in D_Z$ the multiplicity $l_\pi(\xi)$ is finite and in fact*

$$l_\pi(\xi) \leqslant l(\xi).$$

Furthermore the map (4.5.1) *induces a surjection*

$$E \to \pi(U) \tag{4.5.2}$$

of $\mathfrak{g}$-*modules.*

Proof. By Theorem 4.4 it suffices only to observe that the map (4.5.2) is surjective. But this is immediate from the tensor product decomposition (4.4.1) since $\pi(u)$ is a scalar for any $u \in Z$. Q.E.D.

If π is a representation which admits an infinitesimal character and $\xi \in D_Z$ is arbitrary we will say that ξ occurs with full multiplicity in $\pi(U)$ in case $l_\pi(\xi) = l(\xi)$.

4.6. Now return to our finite dimensional module V_λ, $\lambda \in D_Z$. Also let V, as in (4.5.1), be any other (possibly infinite dimensional) U-module and consider the tensor product module $V \otimes V_\lambda$ and the set End $V \otimes V_\lambda$ of all operators on $V \otimes V_\lambda$. The set of all operators on $V \otimes V_\lambda$ which commute with the action of U, (the "commuting ring") will be denoted by S. That is, in a often-used notation,

$$S = \mathrm{End}_{\mathfrak{g}}\, V \otimes V_\lambda . \tag{4.6.1}$$

But now since V_λ is finite dimensional one easily has End $V \otimes V_\lambda =$ End $V \otimes$ End V_λ. Furthermore regarding End $V \otimes$ End V_λ as a tensor product of $\mathfrak{g}$-modules (under the adjoint action) one then has that S is the set (End $V \otimes$ End $V_\lambda)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants. We shall be more concerned with the subring R_π of the commuting ring S defined by putting

$$R_\pi = (\pi(U) \otimes \mathrm{End}\, V_\lambda)^{\mathfrak{g}}. \tag{4.6.2}$$

We will call R_π the strongly commuting ring.

Now let $(\xi_1 ,..., \xi_m) \subseteq D_Z$ be the set of all $\xi \in D_Z$ such that π_ξ occurs with positive multiplicity in End V_λ. Also let $l_j = l_{\pi_\lambda}(\xi_j)$ be the multiplicity of π_{ξ_j} in End V_λ.

Remark 4.6. By Corollary 4.5 one of course has that $l_j \leqslant l(\xi_j)$.

An important integer for us will be the integer r defined by putting

$$r = \sum_{j=1}^{m} l(\xi_j)\, l_j . \tag{4.6.3}$$

For any $\xi \in D_Z$ let $\xi' \in D_Z$ be such that $\pi_{\xi'}$ be equivalent to the representation contragredient to π_ξ .

PROPOSITION 4.6. *Let π, as in (4.5.1), be any representation of U which admits an infinitesimal character. Then the strongly commuting ring R_π is finite dimensional and in fact*

$$\dim R_\pi \leqslant r.$$

Furthermore $\dim R_\pi = r$ *if and only if ξ_j' occurs with full multiplicity in $\pi(U)$ for all $j = 1, 2, \ldots, m$.*

Proof. For any $\xi, \eta \in D_Z$ let $(\operatorname{End} V_\lambda)_\xi$ and $(\pi(U))_\eta$ denote the primary components in $\operatorname{End} V_\lambda$ and $\pi(U)$, respectively, corresponding to π_ξ and π_η . Then $\pi(U) \otimes \operatorname{End} V_\lambda$ is a direct sum of $(\pi(U))_\eta \otimes (\operatorname{End} V_\lambda)_\xi$ over all $\xi, \eta \in D_Z$. Thus R_π is a direct sum of $(\pi(U)_\eta \otimes (\operatorname{End} V_\lambda)_\xi)^{\mathfrak{g}}$ over all $\xi, \eta \in D_Z$. Using Schur's lemma, $((\pi(U))_\eta \otimes (\operatorname{End} V_\lambda)_\xi)^{\mathfrak{g}} = 0$ unless $\eta = \xi'$ and $\xi = \xi_j$ for some $j = 1, 2, \ldots, m$. On the other hand if $W \subseteq (\pi(U))_{\xi_j'}$ and $Y \subseteq (\operatorname{End} V_\lambda)_{\xi_j}$ are irreducible then $(W \otimes Y)^{\mathfrak{g}}$ is one-dimensional, again by Schur's lemma. But since $(\pi(U))_{\xi_j'}$ is a direct sum of $l_\pi(\xi_j')$ subspaces of the form W and $(\operatorname{End} V_\lambda)_{\xi_j}$ is a direct sum of l_j subspaces of the form Y, it follows that

$$\dim(\pi(U)_{\xi_j'} \otimes (\operatorname{End} V_\lambda)_{\xi_j})^{\mathfrak{g}} = l_\pi(\xi_j')\, l_j .$$

Hence R_π is finite dimensional and in fact

$$\dim R_\pi = \sum_{j=1} l_\pi(\xi_j')\, l_j .$$

But by Corollary 4.5 $l_\pi(\xi_j') \leqslant l(\xi_j')$. However, one clearly has $l(\xi') = l(\xi)$ for any $\xi \in D_Z$. Thus $\dim R_\pi \leqslant r$ and equality occurs if and only if $l_\pi(\xi_j') = l(\xi_j')$ for all j. Q.E.D.

4.7. The question as to when $\dim R_\pi = r$ has a particularly nice answer in case π is irreducible and finite dimensional. Recall that d_i , $i = 1, 2, \ldots, k$, are the multiplicities of the weights μ_i occurring in π_λ .

THEOREM 4.7. *Let* $\nu \in D_Z$. *Then* $R_{\pi_\nu} = \mathrm{End}(V_\nu \otimes V_\lambda)$ *is the full commuting ring. Furthermore* $\dim R_{\pi_\nu} \leqslant r$ *and equality occurs if and only if* π_λ *is totally subordinate to* π_ν. *Moreover one has the equation*

$$r = \sum_{i=1}^{k} d_i^2 \tag{4.7.1}$$

and if λ *is totally subordinate to* ν *then there is an isomorphism of algebras*

$$R_{\pi_\nu} \to \bigoplus_{i=1}^{k} M(d_i, C), \tag{4.7.2}$$

where $M(d_i, \mathbb{C})$ *denotes the* $d_1 \times d_i$ *matrix algebra over* $\mathbb{C}$.

In particular there exist $v_0 \in D$ *such that one has an isomorphism for all* $v \in D_Z$ *where* $\nu \geqslant \nu_0$.

Proof. Since V_ν is irreducible one has $\pi_\nu(U) = \mathrm{End}\, V_\nu$ and hence, in the notation of Section 4.6, $R_{\pi_\nu} = S = \mathrm{End}_{\mathfrak{g}}(V_\nu \otimes V_\lambda)$ is the full commuting ring for the action of $\mathfrak{g}$ on $V_\nu \otimes V_\lambda$. Now by Proposition 4.6 one has $\dim R_{\pi_\nu} \leqslant r$. Let

$$\Gamma = \{\nu \in D_Z \mid \dim R_{\pi_\nu} = r\}. \tag{4.7.3}$$

But by Proposition 4.6 one has that if $\nu \in D_Z$ then $\nu \in \Gamma$ if and only if $l_{\pi_\nu}(\xi_j') = l(\xi_j)$, $j = 1, 2, \ldots, m$.

One knows that Γ is not empty since by [5, Lemma 17, p. 403] one has $l_{\pi_\nu}(\xi_j') = l(\xi_j')$, $j = 1, 2, \ldots, m$ for ν sufficiently large. Thus there exists $\nu_1 \in D_Z$ such that for all $\nu \geqslant \nu_1$,

$$\dim R_{\pi_\nu} = r. \tag{4.7.4}$$

But one may view the commuting ring R_{π_ν} from another point of view. Using the notation of Section 4.3, we recall that μ_i, $i = 1, 2, \ldots, k$, are the weights of π_λ and that they occur with multiplicity d_i. But by Theorem 4.3 the $\mathfrak{g}$-irreducible components of $V_\nu \otimes V_\lambda$ are all of the form $\pi_{\nu+\mu_i}$ and this representation occurs with multiplicity $d_i(\nu)$, where $d_i(\nu) \leqslant d_i$. But then one immediately has the existence of an isomorphism

$$R_{\pi_\nu} \to \bigoplus_{i=1}^{k} M(d_i(\nu), \mathbb{C}), \tag{4.7.4}$$

and hence

$$\dim R_{\pi_\nu} = \sum_{i=1}^{k} d_i(\nu)^2. \tag{4.7.5}$$

Thus if $r' = \sum_{i=1}^{k} d_i^2$ one has $\dim R_{\pi_\nu} \leqslant r'$ for all $\nu \in D_Z$ and $\dim R_{\pi_\nu} = r'$ if and only if $d_i = d_i(\nu)$ for $i = 1, 2,..., k$. That is, $\dim R_{\pi_\nu} = r'$ if and only if λ is totally subordinate to ν. In particular, by Proposition 4.1 there exists $\nu_0 \in D_Z$ such that $\dim R_{\pi_\nu} = r'$ for all $\nu \geqslant \nu_0$. But then comparing this with (4.7.4) one must have $r = r'$, establishing (4.7.1), and Γ is the set of all $\nu \in D_Z$ such that π_λ is totally subordinate to ν. The isomorphism (4.7.2) then follows from (4.7.4). The last statement of the theorem follows from Proposition 4.2.
Q.E.D.

4.8. Now let T be the algebra defined by putting

$$T = U \otimes \operatorname{End} V_\lambda, \tag{4.8.1}$$

and let

$$\delta\colon U \to T$$

be the homomorphism defined so that

$$\delta x = x \otimes 1 + 1 \otimes \pi_\lambda(x) \tag{4.8.2}$$

for any $x \in \mathfrak{g}$. This is clearly well defined since $\delta \mid \mathfrak{g}$ is a Lie algebra homomorphism.

Now let $R \subseteq T$ be the commuting algebra of $\delta(U)$ in T. We recall that $Z \subseteq U$ is the center of U so that if we identify U with $U \otimes 1 \subseteq T$ then U and hence Z is a subalgebra of T. Clearly $Z \subseteq R$. Regard R as a Z-module under left multiplication. Since Z is in the center of T it is in the center of R and one could use right multiplication as well.

THEOREM 4.8. *R is a free Z-module of rank r, where r is defined by* (4.6.3).

Proof. By (4.4.1) one has that $T = Z \otimes E \otimes \operatorname{End} V_\lambda$. But then

$$R = Z \otimes ((E \otimes \operatorname{End} V_\lambda) \cap R). \tag{4.8.3}$$

On the other hand, if for any $\xi \in D_Z$, E_ξ denotes the primary π_ξ-component of E as a $\mathfrak{g}$-module under the adjoint representation, then the argument in the proof of Proposition 4.6 also yields that

$$(E \otimes \operatorname{End} V_\lambda) \cap R = \bigoplus_{j=1}^{m} (E_{\xi_j'} \otimes (\operatorname{End} V_\lambda)_{\xi_j}) \cap R. \tag{4.8.4}$$

But since the multiplicity of $\pi_{\xi_j'}$ in $E_{\xi_j'}$ is $l(\xi_j')$ the argument in the proof of Proposition 4.6 also proves that

$$\dim((E_{\xi_j'} \otimes (\mathrm{End}\ V_\lambda)_{\xi_j}) \cap R) = l(\xi_j')\, l_j\,.$$

Hence by (4.8.7), (4.8.4), and (4.6.3) one has that R is a free Z-module of rank r. Q.E.D.

4.9. Now since $\delta(Z)$ centralizes $\delta(U)$, one has

$$\delta(Z) \subseteq R. \tag{4.9.1}$$

Remark 4.9. Note that $\delta(Z)$ lies in the center of R since R by definition centralizes $\delta(U)$.

Now fix $u \in Z$ and let X be an indeterminate. We wish to construct a monic polynomial (which in fact will be the minimal polynomial) $p_u(X) \in Z[X]$ with coefficients in Z so that $p_u(\delta u) = 0$.

Recall (see Section 2.1) that for any σ in the Weyl group W we have associated an element $\tilde{\sigma}$ in the translated Weyl group $\tilde{W}$. The affine map $\tilde{\sigma}$ is not linear but one has

LEMMA 4.9. *Let $\sigma \in W$ and let $\nu, \mu \in \mathfrak{h}'$; then*

$$\tilde{\sigma}(\nu + \mu) = \tilde{\sigma}\nu + \sigma\mu.$$

Proof.

$$\tilde{\sigma}(\nu + \mu) = \sigma(\nu + \mu + \rho) - \rho = (\sigma(\nu + \rho) - \rho) + \sigma\mu = \tilde{\sigma}(\nu) + \sigma\mu. \qquad \text{Q.E.D.}$$

Now recall $\Delta_\lambda = \{\mu_1, \ldots, \mu_k\}$ are the (distinct) weights of π_λ. The set Δ_λ is stable under W. But now if $\mu \in \mathfrak{h}'$ let $g^\mu \in U(\mathfrak{h})$ be defined by

$$g^\mu(\nu) = f_u(\nu + \mu) \tag{4.9.2}$$

for any $\nu \in \mathfrak{h}'$. But $\tilde{W}$ operates on $U(\mathfrak{h})$ and if $\sigma \in W$ we assert that

$$\tilde{\sigma} g^\mu = g^{\sigma\mu}. \tag{4.9.3}$$

Indeed $(\tilde{\sigma} g^\mu)(\nu) = g^\mu(\tilde{\sigma}^{-1}\nu) = f_u(\tilde{\sigma}^{-1}\nu + \mu) = f_u(\tilde{\sigma}^{-1}(\nu + \sigma\mu))$ by Lemma 4.9. But f_u is $\tilde{W}$-invariant (see Theorem 2.2). Thus $(\tilde{\sigma} g^\mu)(\nu) = f_u(\nu + \sigma\mu) = g^{\sigma\mu}(\nu)$, establishing (4.9.3).

But now since $\tilde{W}$ operates on $U(\mathfrak{h})$ it also operates on the polynomial ring $U(\mathfrak{h})[X]$. For any $i = 1, 2, \ldots, k$ let $g_i = g^{\mu_i}$. If $\sigma\mu_i = \mu_j$ then by

(4.9.3) one has $\tilde{\sigma} g_i = g_j$ and hence $\tilde{\sigma}(X - g_i) = X - g_j$. It then follows immediately that if

$$q_u(X) = \prod_{i=1}^{k} (X - g_i), \tag{4.9.4}$$

then $q_u(X) \in U(\mathfrak{h})^{\tilde{W}}[X]$ is a $\tilde{W}$-invariant. That is,

$$q_u(X) = X^k + \sum_{j=1}^{k} f_j X^{k-j}, \tag{4.9.5}$$

where $f_j \in U(h)^{\tilde{W}}$. But now there exists a unique $z_j \in Z$ such that $f_{z_j} = f_j$ by Theorem 2.2.

Put

$$p_u(X) = X^k + \sum_{j=1}^{k} z_j X^{k-j} \tag{4.9.6}$$

so that $p_u(X)$ is monic polynomial of degree k with coefficients in Z. Our main results will follow from

THEOREM 4.9. *Let $u \in Z$ be arbitrary and let $p_u(X)$ be as above. Then $p_u(\delta u) = 0$ in $R \subseteq U \otimes \operatorname{End} V_\lambda = T$.*

Proof. Let s_i, $i = 1, 2, \ldots, r$, be a free Z-basis of R. Write $p_u(\delta u) = \sum_{i=1} h_i s_i$, where $h_i \in Z$. We wish to prove that $h_i = 0$ for any i. To do this it suffices by Theorem 2.2 to show that $f_{h_i} = 0$. But for this it suffices by Propositions 4.1 and 4.2 to show that $f_{h_i}(\nu) = 0$ for all $\nu \in D_Z$ such that π_λ is totally subordinate to π_ν. Let $\nu \in D_Z$ satisfy this condition and let

$$\tilde{\pi}_\nu : T \to \operatorname{End}(V_\nu \otimes V_\lambda) \tag{4.9.7}$$

be the surjective homomorphism defined by putting $\tilde{\pi}_\nu = \pi_\nu \times 1$. One recalls that $T = U \otimes \operatorname{End} V_\lambda$, that

$$\operatorname{End}(V_\nu \otimes V_\lambda) = \operatorname{End} V_\nu \otimes \operatorname{End} V_\lambda,$$

and that $\pi_\nu(U) = \operatorname{End} V_\nu$. The tensor product representation $\pi_\nu \otimes \pi_\lambda : U \to \operatorname{End}(V_\nu \otimes V_\lambda)$ may then be given by

$$\pi_\nu \otimes \pi_\lambda = \tilde{\pi}_\nu \circ \delta. \tag{4.9.8}$$

But now both T and $\operatorname{End}(V_\nu \otimes V_\lambda)$ are completely reducible as $\mathfrak{g}$-modules with respect to the adjoint action. Since $\tilde{\pi}_\nu$ is a surjective

$\mathfrak{g}$-module map it therefore carries invariants onto invariants and hen by restriction $\tilde{\pi}_\nu$ induces a surjective homomorphism

$$\gamma_\nu\colon R \to R_{\pi_\nu} \tag{4.9.}$$

and by Proposition 2.6 one has

$$\gamma_\nu(p_u(\delta u)) = \sum_{i=1}^{r} f_{h_i}(\nu)\, \gamma_\nu(s_i). \tag{4.9.1(}$$

But since Z maps into scalars under γ_ν and since R_{π_ν} is r-dimensiona it follows from the surjectivity of γ_ν that $\gamma_\nu(s_i)$, $i = 1,..., r$, is a basi of R_{π_ν}. Thus to prove $f_{h_i}(\nu) = 0$ it suffices from (4.9.10) to prov that $\gamma_\nu(p_u(\delta u)) = 0$. Put $v = \gamma_\nu(p_u(\delta u))$. From Proposition 2.6 (4.9.6), and (4.9.8) one has

$$v = \tilde{u}^k + \sum_{j=1}^{k} f_{z_j}(\nu)\, \tilde{u}^{k-j},$$

where $\tilde{u} = (\pi_\nu \otimes \pi_\lambda)(u)$. But then by (4.9.4) and (4.9.5) one has

$$v = \prod_{i=1}^{k} (\tilde{u} - f_u(\nu + \mu_i)),$$

since $g_i(\nu) = f_u(\nu + \mu_i)$ (see (4.9.2)). That is, one has

$$v = \prod_{i=1}^{k} (\tilde{u} - \chi_{\nu+\mu_i}(u)).$$

But by Theorem 4.3, $V_\nu \otimes V_\lambda$ as a $\mathfrak{g}$-module decomposes into a direct sum of dim V_λ irreducible submodules any one of which is equivalent to $\pi_{\nu+\mu_i}$ for some i. But $\pi_{\nu+\mu_i}(u)$ reduces to the scalar $\chi_{\nu+\mu_i}(u)$ by Proposition 2.6. Thus $\prod_{i=1}^{k} (\tilde{u} - \chi_{\nu+\mu_i}(u)) = v$ vanishes.

Q.E.D.

5. The Infinitesimal Characters in the Tensor Product $V \otimes V_\lambda$

5.1. Now assume that

$$\pi\colon U \to \operatorname{End} V$$

is any (finite or infinite dimensional) representation which admits an infinitesimal character χ_π (e.g., if π is irreducible). Then by Theorem 2.5 there exists $\nu \in \mathfrak{h}'$ unique up to a transform by an element

$\tilde{\sigma} \in \tilde{W}$ such that $\chi_\pi = \chi_\nu$. Again as above, $\lambda \in D_Z$ is fixed and the set of weights of the finite dimensional representation π_λ is denoted by, $\Delta_\lambda = \{\mu_1, ..., \mu_k\}$. (The sequence μ_i excludes multiplicities.)

Now consider the sequence

$$\chi_{\nu+\mu_1}, \chi_{\nu+\mu_2}, ..., \chi_{\nu+\mu_k} \tag{5.1.1}$$

of infinitesimal characters. We observe that this sequence, apart from the order but including multiplicities, is independent of the representative $\nu \in \tilde{W}\nu$. That is, for any $\sigma \in W$, the sequence $\chi_{\tilde{\sigma}\nu+\mu_1}, ..., \chi_{\tilde{\sigma}\nu+\mu_k}$ is just a permutation of the sequence (5.1.1). This follows from Lemma 4.9 and the fact that Δ_λ is W-invariant.

Now consider the tensor product representation

$$\pi \otimes \pi_\lambda : U \to \text{End } V \otimes V_\lambda .$$

THEOREM 5.1. *Let $u \in Z$ be arbitrary and let $\tilde{u} = (\pi \otimes \pi_\lambda)(u)$. Then the operator $\tilde{u}$ in $V \otimes V_\lambda$ satisfies the equation*

$$\prod_{i=1}^{k} (\tilde{u} - \chi_{\nu+\mu_i}(u)) = 0.$$

Proof. We have only to reverse the argument given in the proof of Theorem 4.9. That is, $\tilde{u} = \tilde{\pi}(\delta u)$, where

$$\tilde{\pi} = \pi \times 1 : T \to \text{End } V \otimes \text{End } V_\lambda = \text{End}(V \otimes V_\lambda).$$

But now, by Theorem 4.9, $p_u(\delta u) = 0$ and hence $\tilde{\pi}(p_u(\delta u)) = 0$. But then by Proposition 2.6,

$$\tilde{\pi}(p_u(\delta u)) = \tilde{u}^k + \sum_{j=1}^{k} f_{z_j}(\nu)\, \tilde{u}^{k-j} = 0. \tag{5.1.2}$$

But then recalling that $g_i(\nu) = f_u(\nu + \mu_i) = \chi_{\nu+\mu_i}(u)$ it follows from (5.1.2) and (4.9.4) that

$$\prod_{i=1}^{k} (\tilde{u} - \chi_{\nu+\mu_i}(u)) = 0. \qquad \text{Q.E.D.}$$

5.2. Put $Y = V \otimes V_\lambda$. If $u \in U$ then since Y is a U-module with respect to $\pi \otimes \pi_\lambda$ we will speak of u rather than $\tilde{u} = (\pi \otimes \pi_\lambda)(u)$ as operating on Y. If $Y_1 \subsetneqq Y_2 \subseteq Y$ and Y_1 and Y_2 are u-stable subspaces then an eigenvalue of u in Y_2/Y_1 will be referred to as a generalized eigenvalue of u in Y. As a consequence of Theorem 5.1 one has

COROLLARY 5.2. *Let $u \in Z$; then an eigenvalue or in fact generalized eigenvalue of u in $V \otimes V_\lambda$ is necessarily of the form $\chi_{\nu+\mu_i}(u)$, for $i = 1, 2, \ldots, k$.*

In particular, if u is the Casimir element, any generalized eigenvalue of u is necessarily of the form $|\nu + \mu_i + \rho|^2 - |\rho|^2$, $i = 1, \ldots, k$. (See Remark 2.3.)

Proof. This is immediate from Theorem 5.1 since if c is a generalized eigenvalue of u we neceessarily have

$$\prod_{i=1}^{k} (c - \chi_{\nu+\mu_i}(u)) = 0. \qquad \text{Q.E.D.}$$

5.3. Rather than dealing with individual elements in Z we can deal with Z all at once. If χ is an infinitesimal character of Z we will say χ occurs in Y if there exist U-sumodules $Y_1 \subsetneqq Y_2 \subseteq Y$ such that the representation of U in Y_2/Y_1 admits an infinitesimal character and χ is that character.

THEOREM 5.3. *Any infinitesimal character χ which occurs in $V \otimes V_\lambda$ is necessarily of the form $\chi_{\nu+\mu_i}$, $i = 1, 2, \ldots, k$.*

Proof. Assume χ is distinct from all the $\chi_{\nu+\mu_i}$ and χ occurs in Y_2/Y_1 in the notation above. But Z is the affine ring on the affine variety Z of all infinitesimal characters of Z. The point χ is distinct from the points $\chi_{\nu+\mu_i}$. Since points are Zariski closed (or by the Chinese remainder theorem) there exists $u \in Z$ such that $\chi(u) \neq 0$ and $\chi_{\nu+\mu_i}(u) = 0$, $i = 1, 2, \ldots, k$. But $\chi(u)$ is a generalized eigenvalue of u in Y. This contradicts Corollary 5.2. Q.E.D.

Remark 5.3. If V is, say, U-irreducible and $\mathfrak{k}$-finitely semisimple then, by Corollary 3.5, $V \otimes V_\lambda$ has a finite composition series and the irreducible components are $\mathfrak{k}$-finitely semisimple. By a Theorem of Harish–Chandra there is only a finite number of such representations with a given infinitesimal character. Theorem 5.3 now limits the irreducible component occurring in $V \otimes V_\lambda$ to the finite ambiguity stated in Harish–Chandra's theorem in that the infinitesimal characters are necessarily of the form $\chi_{\nu+\mu_i}$.

5.4. Among the k characters $\chi_{\nu+\mu_i}$ let n be the number of distinct ones. Order the μ_i so that $\chi_{\nu+\mu_j}$, $j = 1, 2, \ldots, n$, are distinct.

Remark 5.4. Thus for any $1 \leqslant i \leqslant k$ one has $\nu + \mu_i$ is $\tilde{W}$ conjugate to $\nu + \mu_j$ for some unique j, where $1 \leqslant j \leqslant n$.

Let k_j be the multiplicity of $\chi_{\nu+\mu_j}$ among all the $\chi_{\nu+\mu_i}$ so that $\sum_{j=1}^{n} k_j = k$.

Also, for $1 \leqslant j \leqslant n$, let I_j be the kernel of $\chi_{\nu+\mu_j}$ so that I_j is a maximal ideal in Z of codimension 1. If $Y = V \otimes V_\lambda$, put for $1 \leqslant j \leqslant n$ and any i

$$Y_{i,j} = \{y \in Y \mid u \cdot y = 0 \text{ for all } u \in I_j^i\}. \tag{5.4.1}$$

Since I_j is central in U it is clear that $Y_{i,j}$ is a U-submodule of $V \otimes V_\lambda$. Put $Y_j = Y_{k_j,j}$. One then has a filtration

$$0 \leqslant Y_{1,j} \subseteq Y_{2,j} \subseteq \cdots \subseteq Y_{k_j-1,j} \subseteq Y_j \tag{5.4.2}$$

of Y_j by U-submodules.

THEOREM 5.4. *$Y = V \otimes V_\lambda$ is a direct sum of*

$$Y = \bigoplus_{j=1}^{n} Y_j \tag{5.4.3}$$

of the U-submodules Y_j. Furthermore if $X_{i,j} = Y_{i,j}/Y_{i-1,j}$ is not zero then the representation of U in $X_{i,j}$ admits an infinitesimal character and that character is $\chi_{\nu+\mu_j}$.

Proof. It is clear that I_j annihilates $X_{i,j}$ so that if $X_{i,j} \neq 0$ then $X_{i,j}$ admits $\chi_{\nu+\mu_i}$ as an infinitesimal character.

The only thing we have to prove is (5.4.3). Now let $u_1, \ldots, u_l$ be as in Corollary 2.3 so that $Z = C[u_1, \ldots, u_l]$. Let N be the l-dimensional subspace spanned by the u_j. Now the map $u \to \chi_{\nu+\mu_i}(u)$ defines a linear functional l_i on N. One has that $l_i \neq l_j$ for $i \neq j$, $1 \leqslant j \leqslant n$. Indeed if $l_i = l_j$ then $\chi_{\nu+\mu_i} = \chi_{\nu+\mu_j}$ since N generates Z. Let N_0 be the Zariski open set on N on which the function $\prod_{1 \leqslant i < j \leqslant n} (l_i - l_j)$ does not vanish. Fix $u \in N_0$ so that the scalars $\chi_{\nu+\mu_i}(u)$, $i = 1, 2, \ldots, n$, are distinct. But now since u satisfies the polynomial equation

$$\prod_{j=1}^{n} (X - \chi_{\nu+\mu_j}(u))^{k_j}$$

on Y it follows as in the standard finite dimensional argument that if

$$P_j = \{y \in Y \mid (u - \chi_{\nu+\mu_j}(u))^{k_j} y = 0\}, \tag{5.4.4}$$

then

$$Y = \bigoplus_{j=1}^{n} P_j. \tag{5.4.5}$$

It suffices only to show that $P_j = Y_j$. First of all, since u is central P_j is a U-submodule. Now if $v \in N_0$ is linearly dependent on u it is obvious that $(v - \chi_{\nu+\mu_j}(v))^{k_j}$ vanishes on P_j. Assume therefore that $v \in N_0$ is linearly independent of u and that $(v - \chi_{\nu+\mu_j}(v)^{k_j}$ does not vanish on P_j. But since the product of $(v - \chi_{\nu+\mu_i}(v))^{k_i}$ over all $i = 1, 2,\ldots, n$ vanishes on Y and hence on P_j, there exists $i \neq j$, $1 \leqslant i \leqslant n$, such that if $Q_1 = \{y \in P_j \mid vy = \chi_{\nu+\mu_i}(v)y\}$ then $Q_1 \neq 0$. But Q_1 is U-stable and hence u-stable. But clearly the same argument shows that u has an eigenvector in Q_1. But since $Q_1 \subseteq P_j$ the only eigenvalue of u in Q_1 is $\chi_{\nu+\mu_j}(u)$. Thus $Q = \{y \in Q_1 \mid uy = \chi_{\nu+\mu_j}(u)y\}$ is not empty. But since $u, v \in N_0$ the linear function l on the plane B spanned by u and v given by $cu + dv \mapsto c\chi_{\nu+\mu_j}(u) + d\chi_{\nu+\mu_i}(v)$ is not equal to the restriction $l_m \mid B$ for any $m = 1, 2,\ldots, k$. Thus there exists $w \in B$ such that $l(w) \neq \chi_{\nu+\mu_i}(w)$ for any $i = 1, 2,\ldots, k$. But $l(w)$ is an eigenvalue of w in Q and hence in Y. This contradicts Corollary 5.2. This proves that for any $v \in N_0$, $(v - \chi_{\nu+\mu_j}(v))^{k_j}$ vanishes on P_j.

Now obviously $Y_j \subseteq P_j$. On the other hand if N_j is the l-dimensional subspace of Z spanned by $v - \chi_{\nu+\mu_j}(v)$ for all $v \in N$ then I_j is the ideal in Z generated by N_j and $I_j^{k_j}$ is the ideal in Z generated by $N_j^{k_j}$. But by the standard polarization argument $N_j^{k_j}$ is spanned by all elements in Z of the form $(v - \chi_{\nu+\mu_j}(v))^{k_j}$ for all $v \in N$. However, by continuity one can restrict the v to lie in a Zariski open set, e.g., N_0.

Thus $I_j^{k_j}$ is the ideal in Z generated by $(v - \chi_{\nu+\mu_j}(v))^{k_j}$ for all $v \in N_0$. This proves that $I_j^{k_j}$ vanishes on P_j and hence $P_j \subseteq Y_j$, so that $P_j = Y_j$. Q.E.D.

5.5. The nicest case occurs when all the characters $\chi_{\nu+\mu_i}$, $i = 1, 2,\ldots, k$, are distinct. This is the generic situation since there is clearly a Zariski open set $\mathfrak{h}_0' \subseteq \mathfrak{h}'$ such that this is the case if $\nu \in \mathfrak{h}_0'$. Theorem 5.4 reduces to

COROLLARY 5.5. *Assume the character $\chi_{\nu+\mu_i}$, $i = 1, 2,\ldots, k$, are all distinct. Put*

$$Y_0 = \{y \in V \otimes V_\lambda \mid uy = \chi_{\nu+\mu_i}(u)y \text{ for all } u \in Z\} \tag{5.5.1}$$

so that, if not zero, Y_i is the maximal submodule of $V \otimes V_\lambda$ admitting $\chi_{\nu+\mu_i}$ as infinitesimal characters. Then $V \otimes V_\lambda$ is a direct sum

$$V \otimes V_\lambda = \bigoplus_{i=1}^{k} Y_i. \tag{5.5.2}$$

5.6. In the general situation the statement and proof of Theorem 5.4 easily yield

COROLLARY 5.6. *Let $P_0 = 0$ and for $i = 1, 2, \ldots, k$, let*

$$P_i = \left\{ y \in V \otimes V_\lambda \,\middle|\, \left(\prod_{j=1}^{i} (u - \chi_{\nu+\mu_j}(u))\right) y = 0, \text{ for all } u \in Z \right\} \quad (5.6.1)$$

so that

$$0 = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_k = V \otimes V_\lambda .$$

Then if P_i/P_{i-1}, $i = 1, 2, \ldots, k$ is not zero the representation of U on P_i/P_{i-1} admits an infinitesimal character and the character is $\chi_{\nu+\mu_i}$.

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